

# Loop effects and infrared divergences in slow-roll inflation

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## Abstract

Loop corrections to observables in slow-roll inflation are found to diverge no worse than powers of the log of the scale factor, extending Weinberg's theorem to quasi-single field inflation models. Demanding perturbation theory be valid during primordial inflation leads to constraints on the effective lagrangian. This leads to some interesting constraints and coincidences on the landscape of inflationary vacua.

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## I. INTRODUCTION

In recent years there has been much discussion in the literature about quantum effects of long wavelength modes in de Sitter, or slow-roll inflationary backgrounds [1–17]. Depending on the authors, these contributions are negligible, infinite, or somewhere in between. A clear understanding of these issues is therefore important in light of the experimentally verified predictions of the semiclassical inflation theory. Essential to these predictions is the assumption that the dominant contributions to density perturbations are infrared finite, mode-by-mode.

In previous work [18], we emphasized the importance of physical constraints on the choice of initial state and explained how this leads to a theoretical uncertainty in the predictions for the observations of a local observer. For example, in a global de Sitter spacetime, perturbation theory in massive scalar field theory around the Bunch-Davies vacuum appears convergent. Nevertheless it is difficult to explicitly introduce an infrared cutoff, and then remove it maintaining the symmetries. Depending on one’s choice of spacelike slices, such a procedure may be necessary. Moreover once massless fields are included (even the graviton) the procedure of adopting an infrared cutoff appears to fail, and it seems likely the global spacetime is unstable.

On the other hand, for realistic applications to cosmology we are more interested in a local patch of quasi-de Sitter spacetime that expands to our observable universe. In this scenario a comoving infrared cutoff is the simplest accurate model, and most of the questions of principle for global de Sitter become irrelevant [18, 19]. In this context, any sensitivity of observables to the infrared cutoff reflects a genuine theoretical uncertainty in predictions, originating from the lack of a precisely controlled initial state. Such quantum corrections were explored in [18].

In the present work our goal is to extend these results to slow-roll inflation, allowing for the nontrivial time dependence of the Hubble parameter. For models built using scalars with minimal kinetic terms, it is straightforward to combine the physical setup of [18] with the results of Weinberg [1, 2] for this class of models to see that observables at most diverge as a power of a logarithm of the scale factor. While this presents serious problems for the global stability of de Sitter spacetime, the infrared quantum corrections are tiny for primordial slow-roll inflation with realistic parameters.

Xue, Gao and Brandenberger [20] have proposed a related scalar model with non-minimal kinetic terms that evade this conclusion. They find large infrared quantum corrections produce strict bounds on the scalar couplings arising from convergent perturbation theory. In the present work we extend the results of Weinberg to this class of nonminimal kinetic term models, and confirm that at most powers of the logarithm of the scale factor appear in observables. We then re-examine bounds on the couplings by requiring a good perturbative expansion, and find that running of the scalar mass parameters tends to produce an even larger effect than that of the infrared modes, with somewhat less strict bounds emerging than found in [20]. We also show that the infrared corrections in slow roll are bounded above by the corresponding corrections in pure de Sitter spacetime, as one would intuitively expect.

It is interesting to note that these bounds arising from quantum consistency are not far off the kinds of bounds that emerge from tree-level slow-roll considerations, combined with matching the scalar potential to the magnitude of observed density fluctuations [21, 22]. We argue this coincidence may be explained using statistics on a landscape of vacuum states. Thus the saturation of the perturbative bound on the landscape (at least within this class of models) may be regarded a postdiction of the observed density fluctuations. We conclude with a brief discussion of how the late-time instability of a de Sitter region is compatible with the embedding of a de Sitter region in a unitary model for quantum gravity [23, 24] and how the instability timescale that emerges solves the Boltzmann brain paradox of cosmology [25].

## II. IN-IN FORMALISM WITH IR CUTOFF

We consider slow-roll inflation, with an infrared cutoff imposed as in [18]. To obtain a tractable model of slow-roll inflation we consider a quasi-single field inflaton model with two scalars: a slowly rolling inflaton ( $\varphi$ ) and a spectator field ( $\sigma$ ), as already considered in [26] and [20]. Using polar coordinates in field space, the inflaton and spectator correspond to the tangential and radial directions respectively, and the curvature of the inflaton trajectory leads to a minimal coupling between the fields. The action governing the system is[30]

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \left( 1 + \frac{\sigma}{R} \right)^2 g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g^{\mu\nu} \partial_\mu \sigma \partial_\nu \sigma - V(\varphi, \sigma) \right], \quad (1)$$

where  $R$  is a constant and the potential  $V$  will be constrained in such a way that slow-roll conditions for the inflaton are satisfied. We will work in the spatially flat gauge, in which the metric is given by

$$ds^2 = -dt^2 + a(t)^2 d\vec{x}^2, \quad (2)$$

and the scalar metric perturbation has been incorporated in the perturbation of the inflaton field  $\varphi$ . It will also be convenient to use the collective notation

$$\vec{\Phi} = \begin{pmatrix} \varphi \\ \sigma \end{pmatrix}. \quad (3)$$

Background solution: Perturbing the fields via  $\vec{\Phi} = \vec{\Phi}_0 + \delta\vec{\Phi}$ , and then minimizing the action, we find the field equations governing the background solution. Taking the background solution to be spatially homogeneous, the background equations are

$$\ddot{\varphi}_0 + 3H\dot{\varphi}_0 + V'_\varphi = 0, \quad (4)$$

$$\ddot{\sigma}_0 + 3H\dot{\sigma}_0 + V'_\sigma - \frac{\dot{\varphi}_0^2}{R} = 0, \quad (5)$$

where  $H \equiv \frac{\dot{a}}{a}$  is the Hubble parameter and  $V'_\varphi \equiv \partial_\varphi V$  etc. Equation (4) places constraints on the potential for the field  $\varphi_0$  to undergo slow-roll. For the spectator field we pick a constant solution, and without loss of generality we can choose  $\sigma_0 = 0$ . Equation (5) then relates the steepness of the potential in the radial direction to the speed of rolling inflaton by

$$V'_\sigma = \frac{\dot{\varphi}_0^2}{R} \equiv R\lambda(t)^2. \quad (6)$$

Note the slow roll parameter is non-vanishing for non-zero  $\lambda$

$$\epsilon \equiv \frac{m_{pl}^2}{16\pi} \left( \frac{V'}{V} \right)^2 \approx \frac{4\pi\dot{\varphi}_0^2}{H^2 m_{pl}^2} = \frac{4\pi R^2 \lambda^2}{H^2 m_{pl}^2}. \quad (7)$$

### A. The free action

We wish to use in-in formalism to compute two-point correlators of the form

$$G_{ij}(t) = \langle \left( T e^{-i \int_{-\infty_-}^t dt' H_{\text{int}}} \right)^\dagger \delta\Phi_i(t) \delta\Phi_j(t) \left( T e^{-i \int_{-\infty_+}^t dt'' H_{\text{int}}} \right) \rangle, \quad (8)$$

where the mode functions  $\delta\Phi$  are determined by the free part of the Hamiltonian, and the interaction part is taken into account perturbatively as in (8). Thus we need the free

Hamiltonian, which is defined as the part quadratic in perturbations [1, 27]. Expanding the action around the background solution  $\Phi_0$  up to second order yields

$$S_{\text{free}} = S_0 + \frac{1}{2} \int d^4x \sqrt{-g} \left[ (\dot{\delta}\varphi)^2 - a^{-2}(\nabla\delta\varphi)^2 - V''_{\varphi\varphi}(\delta\varphi)^2 + (\dot{\delta}\sigma)^2 - a^{-2}(\nabla\delta\sigma)^2 - (V''_{\sigma\sigma} - \lambda^2)(\delta\sigma)^2 + 4\lambda\delta\sigma(\dot{\delta}\varphi) - 2V''_{\sigma\varphi}\delta\sigma\delta\varphi \right] \quad (9)$$

$$= S_0 + \frac{1}{2} \int d^3\vec{k} dt \sqrt{-g} \left[ \dot{\varphi}_k^2 - \left( \frac{k^2}{a^2} + V''_{\varphi\varphi} \right) \varphi_k^2 + \dot{\sigma}_k^2 - \left( \frac{k^2}{a^2} + V''_{\sigma\sigma} - \lambda^2 \right) \sigma_k^2 + 4\lambda\sigma_k\dot{\varphi}_k - 2V''_{\sigma\varphi}\sigma_k\varphi_k \right], \quad (10)$$

where  $S_0 \equiv \int d^4x \sqrt{-g} [\frac{1}{2}\dot{\varphi}_0^2 - V(\varphi_0, \sigma_0)]$  contains the zeroth-order terms, and we have switched to momentum space via

$$\delta\vec{\Phi}(t, \vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^{\frac{3}{2}}} e^{i\vec{k}\cdot\vec{x}} \vec{\Phi}_k(t). \quad (11)$$

Note that in Fourier space we drop the  $\delta$  in front of the perturbation.

The field equations: From (10) one can derive the field equations

$$\ddot{\varphi}_k + 3H\dot{\varphi}_k + \left( \frac{k^2}{a^2} + V''_{\varphi\varphi} \right) \varphi_k = -2\partial_t(\lambda\sigma_k) - 6H\lambda\sigma_k - V''_{\sigma\varphi}\sigma_k, \quad (12)$$

$$\ddot{\sigma}_k + 3H\dot{\sigma}_k + \left( \frac{k^2}{a^2} + V''_{\sigma\sigma} - \lambda^2 \right) \sigma_k = 2\lambda\dot{\varphi}_k. \quad (13)$$

We take the inflaton to be massless ( $V''_{\varphi\varphi} = 0$ ), and denote by  $m^2 \equiv V''_{\sigma\sigma} - \lambda^2$  the effective ‘mass’ of the spectator field  $\sigma$ . We also take the potential to be of the form  $V = V(\varphi) + V(\sigma)$  to leading order, implying  $V''_{\varphi\sigma} = 0$ . Both of these constraints are consistent with the analysis of [20, 26].

In order to have a solvable system, from now on we will also take the inflaton to roll at a constant speed, so  $\dot{\lambda} = 0$ . Then (5) tells us that the potential has to be chosen such that the slope  $V'_\sigma$  is constant along the trajectory. This departs from the analysis of [20, 26], who make no such assumption. We choose to set this constraint, because a central tenet of this article is that in order to compute correlators of type (8) one has to treat  $\lambda$  analytically in an exact manner, as opposed to perturbatively. In [20, 26] the cross-term  $\lambda\sigma\dot{\varphi}$  is bundled into the interaction Hamiltonian, whereas we treat it as a part of  $H_{\text{free}}$ , and restrict to constant  $\lambda$  in order to be able to explicitly solve the field equations.

Finally, at the level of the field equations we will work in an ‘instantaneously de Sitter’ approximation, in which the scale factor is given by  $a(t) = \exp(Ht)$ , with a constant  $H$ .

This approximation is valid over time scales

$$H^2\epsilon\Delta t \ll H \implies \Delta t \ll \frac{Hm_{pl}^2}{4\pi R^2\lambda^2}$$

which is quite sufficient for our purposes. In particular, it can contain the regime where effects nonperturbative in  $\lambda\Delta t$  become important. Such effects are dropped in [20, 26] where a perturbative expansion is  $\lambda$  is considered.

At this point it is also convenient to switch to conformal time, defined by

$$dt = a(t)d\tau, \quad \Rightarrow \quad \tau = \int \frac{dt}{a(t)} = -\frac{1}{Ha}. \quad (14)$$

Incorporating the constraints and approximations the field equations in conformal time become

$$\varphi_k'' - \frac{2}{\tau}\varphi_k' + k^2\varphi_k = \frac{2\lambda}{H\tau} \left( \sigma_k' - \frac{3}{\tau}\sigma_k \right), \quad (15)$$

$$\sigma_k'' - \frac{2}{\tau}\sigma_k' + \left( k^2 + \frac{m^2}{H^2\tau^2} \right) \sigma_k = -\frac{2\lambda}{H\tau} \varphi_k', \quad (16)$$

where  $' \equiv \partial_\tau$ .

## B. The free field solution

We will now solve the field equations (15,16) perturbatively in  $k$  using the Green's function method. Using the expansion

$$\vec{\Phi} = \sum_{i=0}^{\infty} k^{2i} \vec{\Phi}_i, \quad (17)$$

we can write the field equations in matrix notation as

$$L\vec{\Phi}_i = -k^2\vec{\Phi}_{i-1}, \quad \text{with} \quad L \equiv \begin{pmatrix} \partial_\tau^2 - \frac{2}{\tau}\partial_\tau & -\frac{2\lambda}{H\tau} \left( \partial_\tau - \frac{3}{\tau} \right) \\ \frac{2\lambda}{H\tau}\partial_\tau & \partial_\tau^2 - \frac{2}{\tau}\partial_\tau + \frac{m^2}{H^2\tau^2} \end{pmatrix}. \quad (18)$$

Note that one should not confuse the mode function  $\vec{\Phi}_{i=0}$  with the values of the background fields  $\vec{\Phi}_0$  found earlier. From now on the background values will only appear inside  $\lambda = \dot{\varphi}_0/R$ , so no confusion should arise.

We can easily solve  $\vec{\Phi}_0$  from  $L\vec{\Phi}_0 = 0$ , which has power law solutions. One verifies that

the general solution is

$$\vec{\Phi}_0 = \sum_{i=1}^4 \vec{A}_i \left( \frac{\tau}{\tau_0} \right)^{\alpha_i}, \quad \text{with } \vec{a} = (0, 3, \alpha_-, \alpha_+), \quad \vec{A}_i = \begin{pmatrix} a_i^\varphi \\ a_i^\sigma \end{pmatrix}, \quad (19)$$

$$\alpha_\pm = \frac{3}{2} \left( 1 \pm \sqrt{1 - \left( \frac{2m}{3H} \right)^2 - \left( \frac{4\lambda}{3H} \right)^2} \right),$$

$$\vec{a}^\sigma = \left( 0, -\frac{6\lambda H}{m^2} a_2^\varphi, \frac{3 + \alpha_- - \alpha_+}{4\lambda/H} a_3^\varphi, \frac{3 - \alpha_- + \alpha_+}{4\lambda/H} a_4^\varphi \right),$$

where  $\tau_0$  is a fixed initial time. We also need the Green's function, defined by

$$LG(\tau, \tau') = \delta(\tau - \tau') \mathbb{1}, \quad \text{with } G = \begin{pmatrix} G_{\varphi\varphi} & G_{\varphi\sigma} \\ G_{\sigma\varphi} & G_{\sigma\sigma} \end{pmatrix}. \quad (20)$$

We relegate the computation of  $G$  into appendix B, here we only present the result

$$G(\tau, \tau') = \sum_{i=1}^4 \bar{C}_i \Theta(\tau' - \tau) \tau' \left( \frac{\tau}{\tau'} \right)^{\alpha_i}, \quad (21)$$

where the  $\bar{C}_i$  are constant matrices explicitly given in (B6).

We can now use the Green's function iteratively to solve for higher orders  $\Phi_i$ . We have

$$\vec{\Phi}_1(\tau) = -k^2 \int_{\tau_0}^0 d\tau' G(\tau, \tau') \cdot \vec{\Phi}_0(\tau') = (k\tau_0)^2 \sum_{i,j=1}^4 \frac{\bar{C}_i \cdot \vec{A}_j}{2 - \alpha_i + \alpha_j} \tau_0^{\alpha_j} \left( \frac{\tau}{\tau_0} \right)^{2+\alpha_j}. \quad (22)$$

Late times: The four independent solutions at late-time take the form

$$\vec{\Phi}(\tau) = a_0^\varphi \left( 1 + (k\tau_0)^2 a_{0,1}^\varphi \left( \frac{\tau}{\tau_0} \right)^2 + \dots \right) + a_1^\varphi \left( \left( \frac{\tau}{\tau_0} \right)^3 + (k\tau_0)^2 a_{0,1}^\varphi \left( \frac{\tau}{\tau_0} \right)^5 + \dots \right)$$

$$+ a_2^\varphi \left( \left( \frac{\tau}{\tau_0} \right)^{\alpha_-} + (k\tau_0)^2 a_{2,1}^\varphi \left( \frac{\tau}{\tau_0} \right)^{\alpha_-+2} + \dots \right)$$

$$+ a_3^\varphi \left( \left( \frac{\tau}{\tau_0} \right)^{\alpha_+} + (k\tau_0)^2 a_{3,1}^\varphi \left( \frac{\tau}{\tau_0} \right)^{\alpha_++2} + \dots \right) \quad (23)$$

$$\frac{3+\alpha_--\alpha_+}{4\lambda/H} \left( \frac{\tau}{\tau_0} \right)^{\alpha_-} + (k\tau_0)^2 a_{2,1}^\sigma \left( \frac{\tau}{\tau_0} \right)^{\alpha_-+2} + \dots$$

$$\frac{3-\alpha_-+\alpha_+}{4\lambda/H} \left( \frac{\tau}{\tau_0} \right)^{\alpha_+} + (k\tau_0)^2 a_{3,1}^\sigma \left( \frac{\tau}{\tau_0} \right)^{\alpha_++2} + \dots$$

where the new coefficients  $a_{i,1}^\varphi$  and  $a_{i,1}^\sigma$  are functions only of  $m, \lambda$  and may be read-off from (22). The terms  $\dots$  denote subleading terms as  $\tau \rightarrow 0$ .

### C. Quantization

We quantize these fields using the mode expansion

$$\Phi(x, t) = \int d^3q e^{iq \cdot x} \Phi(q, t) \cdot \boldsymbol{\alpha}(q) + e^{-iq \cdot x} \Phi^*(q, t) \cdot \boldsymbol{\alpha}^*(q),$$

where

$$[\alpha_i(q), \alpha_j^*(q')] = \delta_{ij} \delta^3(q - q'), \quad [\alpha_i(q), \alpha_j(q')] = 0,$$

and take the vacuum to be the Bunch-Davies vacuum at the start of inflation, annihilated by these annihilation operators. In this early time limit, the modes of interest (recall the comoving infrared cutoff) are all inside the horizon, and oscillate with time. The quantization proceeds in the standard way.

Later we will need to also estimate the commutator of the fields in the late-time limit, where the fields asymptote to the form (23). Now the modes of interest are far outside the horizon where they decay as real powers of the scale factor (23). At leading order as  $\tau \rightarrow 0$  (late times),

$$\Phi(q, t) = \begin{pmatrix} C_q \tau^0 + D_q \tau^3 & E_q \tau^{\alpha_-} + F_q \tau^{\alpha_+} \\ -\frac{6\lambda H}{m^2} D_q \tau^3 & \frac{3+\alpha_- - \alpha_+}{4\lambda/H} E_q \tau^{\alpha_-} + \frac{3-\alpha_- + \alpha_+}{4\lambda/H} F_q \tau^{\alpha_+} \end{pmatrix}$$

where the complex coefficients  $C_q, D_q, E_q, F_q$  are fixed by matching to the early time modes at horizon crossing. When we compute the commutator  $[\Phi_i(x, t), \Phi_j(x', t')]$  only cross terms between the pairs  $\tau^0, \tau^3$  and  $\tau^{\alpha_-}, \tau^{\alpha_+}$  survive, so the commutator falls off as  $\tau^3$  as  $\tau \rightarrow 0$ .

### D. Late-time limit of observables

We follow Weinberg [1, 2] when computing the leading late-time terms. As emphasized in his work, there are delicate cancellations between terms, which are most easily taken care of using the commutator expression

$$\langle Q(t) \rangle = \sum_{N=0}^{\infty} i^N \int_{t_0}^t dt_N \int_{t_0}^{t_N} dt_{N-1} \cdots \int_{t_0}^{t_2} dt_1 \langle [H_I(t_1), [H_I(t_2), \cdots [H_I(t_N), Q_I(t)] \cdots]] \rangle \tag{24}$$

where the subscript  $I$  denotes interaction picture operators. Note that rather than taking  $t_0 \rightarrow -\infty$  as in [1, 2], we keep it finite, as part of the procedure introduced in [18] for keeping track of the effect of the initial state on observables. Weinberg investigated the

leading late-time divergences for massless and massive minimally coupled scalar fields, as well as Dirac particles and vector particles. Here we will apply this approach to quasi-single field inflation.

As argued above, the commutator of any pair of elementary fields falls off as  $\tau^3$  or  $a(t)^{-3}$ . The same will also be true if one considers time or space derivatives of these fields. The interaction hamiltonian  $H_I$  will be built of products of such fields and derivatives, and contain a volume derivative going as  $a(t)^3$ . As is apparent from (24) the  $H_I$  will always appear inside a commutator, so these factors of  $a(t)$  will cancel.

In general observable we may also encounter additional powers of fields that are not inside commutators. These introduce at worst constant factors (if only  $\varphi(x, t)$  appears in  $Q(t)$ ) or factors that fall at least as fast as  $\tau^{\alpha_-}$ , provided some factors of  $\sigma(x, t)$  appear, or provided sufficient inverse powers of  $a(t)$  appear in derivatives.

We conclude then that observables built solely out of products of  $\varphi(x, t)$  do have late time divergences, arising from the time integrals in (24). These lead to divergences as powers of  $\log \tau$ . These divergences are qualitatively the same as the case of an interacting massless minimally coupled scalar considered in [18]. Of course, in order for inflation to end, this scalar must acquire a mass, which provides a natural physical cutoff to these time integrals. Estimates made in [18] (in the models considered there) show these loop effects are negligible for primordial inflation compared to the tree-level contributions, and only become significant in genuine asymptotic future de Sitter phases – where they are capable of driving an instability. Nevertheless there exists a large class of infrared finite slow-roll observables for quasi-single field inflation which contain either derivatives of  $\varphi(x, t)$  or factors of  $\sigma(x, t)$ .

### III. LOOP CORRECTIONS AND RENORMALIZATION: INFRARED AND UV CONTRIBUTIONS

The authors of [20] considered a closely related set of questions, and argued much larger infrared terms appear due to slow-roll effects. They find divergences that go like powers of  $(\frac{H_{initial}}{H})^2/\epsilon$  (with  $\epsilon$  the slow-roll parameter (7),  $H$  the late-time Hubble parameter, and  $H_{initial}$  the Hubble parameter at the start of inflation). Here we will show these conclusions change when UV divergences are treated with a physical renormalization prescription. The apparent divergences as  $\epsilon \rightarrow 0$  disappear, but are replaced by late time divergences involv-

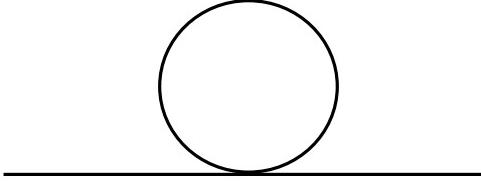


Figure 1: Infrared divergent diagram

ing powers of  $\log a(t)$ . Such divergences match those expected from the de Sitter limit. Demanding that these late-time loop corrections not destroy perturbation theory leads to constraints on the interaction potential, which take a similar form to those argued for in [20], but differ in the details.

In section III.1 of [20] they consider loop corrections to the scalar curvature perturbation two-point function (we refer to [20] for details), due to a massless (or sufficiently light) entropy perturbation. This can be represented by the  $\sigma$  field of the model previously discussed, with a higher order  $g\sigma^4$  self-coupling included. The result of their analysis is that the leading IR divergences comes from a subdiagram involving the  $\sigma$  self-interaction . The corrections appears in eqn (56) [20]. Carrying over their result for the subdiagram, and also including a physical UV cutoff we obtain

$$g \int_{\Lambda_{IR}}^{\Lambda_{IR,phys}a(t)} \frac{dq}{q} H_{initial}^2 \left( \frac{q}{k_{initial}} \right)^{-2\epsilon} + g \int_{\Lambda_{IR,phys}}^{\Lambda_{UV,phys}} \frac{d^3 q}{q^2} = \frac{g H_{initial}^2}{2\epsilon} \left( 1 - e^{-2\epsilon H_{initial} t} \left( \frac{\Lambda_{IR,phys}}{H_{initial}} \right)^{-2\epsilon} \right) + g (\Lambda_{UV,phys}^2 - \Lambda_{IR,phys}^2), \quad (25)$$

where we have estimated the UV and IR divergent terms by breaking the range of integration up at a physical intermediate scale, and used the appropriate asymptotic forms of the propagator. Note a comoving IR cutoff and a proper UV cutoff is used as in [18]. The computation of [20] also uses a comoving IR cutoff, but are less explicit about their choice of UV cutoff. A mass counterterm must be chosen to cancel the UV divergence, and impose a renormalization condition. This is described in more detail in appendix A. The result is that the would-be  $H_{initial}^2/\epsilon$  divergence disappears when a physical renormalization prescription is imposed for the mass of the  $\sigma$  field.

It is also worth pointing out that even the long wavelength contribution to (25) is bounded from above by the pure de Sitter result, where  $\epsilon = 0$ . This follows by choosing the scale  $k_{initial} = \Lambda_{IR} = \mathcal{O}(H_{initial})$ , and noting for slow-roll with  $\epsilon > 0$ , the integrand is always positive and less than the pure de Sitter answer ( $\epsilon = 0$ ) throughout the range of integration.

This point is at odds with the answer obtained in [20], which may be traced to them dropping all but the first term on the right-hand side of (25).

An important consistency condition is demanding that the perturbative expansion in  $g$  converge. In analyzing this question, we allow for the mass renormalization of the inflaton to incorporate generic effects of new physics near the GUT scale, and still require the perturbative expansion be valid. The implications for the one-loop renormalization are described in appendix A. Let us examine the physical constraints that emerge from this. If we insert this subdiagram into the full expression for the one loop correction to the two-point function [18] (simply working in the  $g\sigma^4$  sector of the theory), and ask when perturbation theory is valid, we obtain the condition at the end of inflation that

$$gH^2 \left( N^3 + N^2 \frac{\Lambda_{GUT}^2}{H^2} \right) \ll H^2(N - \log(-\Lambda_{IR}\tau_0)).$$

Now the  $N^3$  term on the left yields a constraint  $g \ll 1/N^2 \sim 10^{-4}$  for massless perturbation theory to be valid. The other term requires

$$gN \frac{\Lambda_{GUT}^2}{H^2} \ll 1.$$

Let us put in some typical values assuming we wish to use the field theory for the inflaton from a UV cutoff near the GUT scale (beyond which we expect new physics to set in), down to scales below the Hubble scale. We set  $H_{initial} = 10^{14} GeV$  and  $\Lambda_{GUT} = 10^{16} GeV$ . This yields

$$g \ll 10^{-6}.$$

So we see while these effects are compatible with the bounds of [20] the bounds found there are not the whole story, and stronger constraints emerge from the consideration of typical UV effects due to renormalization. Another difference with [20] is the powers of  $N$ . In both our work, and [20] a comoving infrared cutoff is used. The justification for this is elaborated in [18]. However the computation of [20] appears to use estimates for amplitudes obtained using the formalism of [8], who instead use a physical/proper distance infrared cutoff.

Finally we can estimate the two loop contribution coming from the diagram shown in figure 2, arising from the coupling of the scalar curvature to the scalar field as described in [20]. Following the same type of computation as above, we find a constraint of the form

$$gH^2 \left( N^4 + N^3 \left( \frac{\Lambda_{GUT}^2}{H^2} \right) + N^2 \left( \frac{\Lambda_{GUT}^2}{H^2} \right)^2 \right) \ll H^2(N - \log(-\Lambda_{IR}\tau_0)).$$

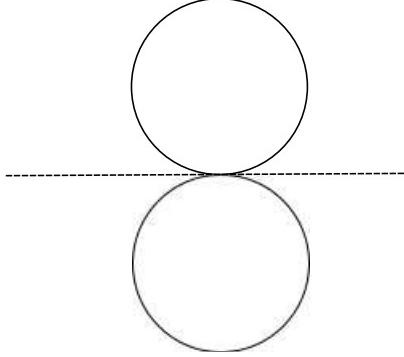


Figure 2: Sub-diagram with two scalar curvature external lines, and the scalar field lines fully contracted.

This then yields the dominant condition

$$gN \frac{\Lambda_{GUT}^4}{H^4} \ll 1, \quad (26)$$

so that  $g \ll 10^{-10}$  which numerically is comparable with the bound of [20], though the dominant effect is the short distance renormalization of the scalar mass, rather than large distance slow-roll terms.

#### IV. COMMENTS ON THE LANDSCAPE

If we had considered single-field inflation with a potential  $g\sigma^4$ , a tree-level bound on  $g$  emerges by matching the observed  $\delta\rho/\rho \sim 10^{-5}$  with the value predicted by slow-roll [21, 22]. This yields  $\delta\rho/\rho \gtrsim Ng^{1/2}$  so that  $g < 10^{-13}$ . It is interesting to point out that the loop-level bound (26) is comparable with this tree-level bound. This coincidence suggests an anthropic relation. Namely anthropic/landscape considerations would tend to statistically favor the largest value of  $g$  compatible with the basic physics of the model. To make  $g$  exceed our quantum bound requires new physics to appear before the GUT scale, taking us out of this class of model. Taking  $g$  to saturate the bound, and assuming for the sake of argument that one is restricted to working with this family of scalar models, one is then led to a postdiction for  $\delta\rho/\rho$  matching observation.[31]

Finally, it is also worth commenting further on the gravitational version of the late-time instability of de Sitter spacetime found in [18] due to the choice of graviton initial state. It was argued there that future eternal de Sitter is actually unstable on a timescale of  $10^{122}$

e-folds. It has already been noted that some kind of late time instability for de Sitter is needed for compatibility with the class of unitary models for quantum de Sitter regions, considered in [23, 24, 28]. There is was pointed out that this can solve the proliferation of Boltzmann brain observers (see [25] for background material).

In the present context, we find the timescale associated with the production of a Boltzmann brain along the path of some timelike geodesic in an expanding universe to be of order its inverse Boltzmann factor  $e^{E/kT} \approx e^{10^{65}}$  for an observer of order 1 mole of protons, with  $T$  the temperature of the present cosmological horizon. This timescale is much larger than the above instability timescale. The infrared instability of de Sitter spacetime thus has a chance to restore our status as typical observers, solving one of the many problems associated with doing statistics on a landscape of theory vacua.

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### Appendix A: Renormalization in cosmological spacetimes

It is helpful to review renormalization in the context of the models considered here, and in [18], filling in some additional details omitted in the earlier work. The scalar coupling  $\lambda$  of [18] will be replaced by  $g$  here, to avoid confusion with the discussion in Section 2. The loop diagram figure 1 gives rise to the integral of eqn. (9) of [18]. The potentially divergent terms arise from the IR and UV ends of the integral, and take the form

$$\begin{aligned} L(\tau_v) &= \frac{-ig}{(2\pi)^2 H^2 \tau_v^4} \left( \tau_v^{2\gamma} \int_{\Lambda_{IR}}^{-\tau_v^{-1}} dp p^{-1+2\gamma} + \tau_v^2 \int_{-\tau_v^{-1}}^{\Lambda_{UV} a(\tau_v)} dp \frac{p^2}{\sqrt{p^2 + m^2 a(\tau_v)^2}} \right) \\ &= \frac{-ig}{(2\pi)^2 H^2 \tau_v^4} \left( \frac{1 - (-\Lambda_{IR} \tau_v)^{2\gamma}}{2\gamma} + \frac{1}{2} \left( \frac{\Lambda_{UV}}{H} \right)^2 - \frac{1}{2} \left( \frac{m}{H} \right)^2 \log \frac{\Lambda_{UV}}{m} \right), \end{aligned}$$

with  $\gamma = m^2/(3H^2)$ , and  $\tau_v$  the conformal time of the vertex factor insertion. Here we have included the subleading log UV divergent term omitted in [18], and assumed  $\Lambda_{UV} \gg m$ .

Now let us consider choosing the counter-term so that a physical mass renormalization condition is imposed at some scale  $\mu$ . In keeping with the small mass/early time expansion

used in this paper, we expand the IR term in powers of  $m^2$  to get

$$L(\tau_v) = \frac{ig}{2(2\pi)^2 H^2 \tau_v^4} \left( \log(-\Lambda_{IR} \tau_v) - \left( \frac{\Lambda_{UV}}{H} \right)^2 + \left( \frac{m}{H} \right)^2 \log \frac{\Lambda_{UV}}{m} \right).$$

In this way, we see that with a comoving infrared cutoff, we cannot completely eliminate the IR divergence into a mass renormalization, due to the additional  $\log \tau_v$  dependence, which causes problems at very late times. With a proper IR cutoff such a renormalization is possible, as discussed in [8], however as discussed in [18] such proper IR cutoffs are unphysical for spacetimes of cosmological interest (an example being bubble walls moving faster than the speed of light).

Now a mass counter-term produces a shift

$$\delta L(\tau_v) = -\frac{i}{H^4 \tau_v^4} \delta m^2,$$

so comparing the UV divergent terms with what we usually have with flat spacetime renormalization we choose to impose the renormalization condition

$$\delta m^2 + \frac{g}{8\pi^2} \left( \Lambda_{UV}^2 - m^2 \log \frac{\Lambda_{UV}}{m} - H^2 \log \Lambda_{IR} \right) = m_{phys}^2.$$

Now we wish to impose the renormalization group equation  $\Lambda_{UV} dm_{phys}^2 / d\Lambda_{UV} = 0$  and view  $m^2 + \delta m^2$  as the bare mass squared  $m_0^2$ , which implies

$$\Lambda_{UV} \frac{dm_0^2}{d\Lambda_{UV}} + \frac{g}{8\pi^2} (2\Lambda_{UV}^2 - m_0^2) = 0.$$

Integrating this equation we find

$$m_0^2(\Lambda_{UV}) = \frac{g}{8\pi} (m_0^2 \log \Lambda_{UV} - \Lambda_{UV}^2) + c,$$

where  $c$  is a constant independent of  $\Lambda_{UV}$  to be fixed by the renormalization condition. Substituting we find the physical mass

$$m_{phys}^2 = c + \frac{g}{8\pi^2} (-H^2 \log \Lambda_{IR} + m_0^2 \log m_{phys}),$$

therefore we fix

$$c = m_{phys}^2 - \frac{g}{8\pi^2} (-H^2 \log \Lambda_{IR} + m_{phys}^2 \log m_{phys}),$$

at leading order in  $g$ . Thus the loop diagram with mass counterterm gives

$$L(\tau_v) = \frac{ig}{2(2\pi)^2 H^2 \tau_v^4} \log(-\tau_v),$$

in this light scalar/early time approximation, and all dependence on the UV and IR cutoffs disappears, with the exception of the  $\log(-\tau_v)$  term which comes from the choice of comoving IR cutoff. Note that additional dependence on the IR cutoff appears when the time integrals of the in-in formulation are performed, as elaborated in [18].

Finally, it is useful to reconsider the above computation, assuming instead that new physics sets in at some high physical scale  $\Lambda_{GUT} > H$ . For example, a new field  $\varphi$  of mass  $\Lambda_{GUT}$  coupling via  $g\sigma^2\varphi^2$ . In this case we do indeed find a correction of the form

$$L(\tau_v) = \frac{i\lambda}{2(2\pi)^2 H^2 \tau_v^4} \left( \log(-\tau_v) - \left( \frac{\Lambda_{GUT}}{H} \right)^2 \right),$$

showing new physics does indeed lead to a quadratic shift in the mass. This form will be useful for estimating the range of the perturbative validity of slow roll theory for energy scales approaching the GUT scale.

## Appendix B: Green's Function

In this appendix we solve equation (20) to derive the Green's function of the system. Since we know the zeroth order solution (19), a good ansatz is

$$G(\tau, \tau') = \sum_{k=1}^4 C_k \Theta(\tau' - \tau) \tau'^{\alpha_k}, \quad \text{with} \quad C_k = \begin{pmatrix} c_{11,k} & c_{12,k} \\ c_{21,k} & c_{22,k} \end{pmatrix}, \quad (\text{B1})$$

where the coefficients  $c_{ij,k}$  satisfy the same relations as  $a_k^{\varphi,\sigma}$ , i.e.

$$\frac{a_k^\sigma}{a_k^\varphi} = \frac{c_{21,k}}{c_{11,k}} = \frac{c_{22,k}}{c_{12,k}}. \quad (\text{B2})$$

In order to fix the rest of the coefficients  $c_{ij,k}$  we integrate (20) over the range  $\tau \in [\tau' - \epsilon, \tau' + \epsilon]$ , computing to order  $\mathcal{O}(\epsilon^0)$ ,

$$1 = \int_{\tau' - \epsilon}^{\tau' + \epsilon} d\tau L \Theta(\tau - \tau') C_k \tau'^{\alpha_k} = - \int_{\tau' - \epsilon}^{\tau' + \epsilon} d\tau (C_k \tau'^{\alpha_k}) \frac{\delta(\tau - \tau')}{\tau - \tau'} + 2\tau'^{\alpha_k} \begin{pmatrix} \alpha_k - 1 & -\frac{\lambda}{H} \\ \frac{\lambda}{H} & \alpha_k - 1 \end{pmatrix} \cdot C_k, \quad (\text{B3})$$

where summing over repeated indices is implied. The first term arises from  $\partial_\tau^2 \Theta(\tau' - \tau)$  and is potentially divergent; we need to demand

$$C_k \tau'^{\alpha_k} = 0 \quad (\text{B4})$$

for it to vanish. The second term implies

$$2\tau'^{\alpha_k} \begin{pmatrix} \alpha_k - 1 & \frac{\lambda}{H} \\ -\frac{\lambda}{H} & \alpha_k - 1 \end{pmatrix} \cdot C_k = \mathbb{1}. \quad (\text{B5})$$

Expressing  $c_{2j,k}$  in terms of  $c_{1j,k}$  using (B2) leaves us with eight unfixed coefficients ( $c_{1j,k}$ ). The remaining constraints (B4) and (B5) do not mix  $c_{11,k}$  and  $c_{12,k}$ , and hence we are left with two groups of four unfixed coefficients, with four constraints for each group. Hence solving for  $c_{ij,k}$  amounts to inverting  $4 \times 4$  matrices, and we find the coefficients to be given by

$$\begin{aligned} C_k &\equiv \bar{C}_k \tau'^{1-\alpha_k}, \\ \bar{C}_1 &= \begin{pmatrix} -\frac{m^2}{6(m^2+4\lambda^2)} & \frac{\lambda}{m^2+4\lambda^2} \\ 0 & 0 \end{pmatrix}, \quad \bar{C}_2 = \begin{pmatrix} \frac{m^2}{6(m^2+4\lambda^2)} & 0 \\ -\frac{\lambda}{m^2+4\lambda^2} & 0 \end{pmatrix}, \\ \bar{C}_3 &= - \begin{pmatrix} \frac{2\lambda^2}{(m^2+4\lambda^2)\sqrt{9-4m^2-16\lambda^2}} & \frac{2\lambda}{4m^2+16\lambda^2-9+3\sqrt{9-4m^2+16\lambda^2}} \\ \frac{\lambda(3-\sqrt{9-4m^2-16\lambda^2})}{2(m^2+4\lambda^2)\sqrt{9-4m^2-16\lambda^2}} & \frac{1}{2\sqrt{9-4m^2-16\lambda^2}} \end{pmatrix}, \\ \bar{C}_4 &= \begin{pmatrix} \frac{2\lambda^2}{(m^2+4\lambda^2)\sqrt{9-4m^2-16\lambda^2}} & \frac{2\lambda}{9-4m^2-16\lambda^2+3\sqrt{9-4m^2+16\lambda^2}} \\ \frac{\lambda(3+\sqrt{9-4m^2-16\lambda^2})}{2(m^2+4\lambda^2)\sqrt{9-4m^2-16\lambda^2}} & \frac{1}{2\sqrt{9-4m^2-16\lambda^2}} \end{pmatrix}, \end{aligned} \quad (\text{B6})$$

where we set  $H = 1$ ; it can be restored by scaling  $m \rightarrow \frac{m}{H}$  and  $\lambda \rightarrow \frac{\lambda}{H}$ .

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- [30] This action differs from the one considered in [26] in that our inflaton has been scaled by  $R$  for later convenience,  $\varphi_{\text{us}} = R\theta_{\text{them}}$ . Note that the kinetic term for  $\theta$  would be  $-\frac{1}{2}(R + \sigma)^2 g^{\mu\nu} \partial_\mu \theta \partial_\nu \theta$ , as expected when  $\theta$  is the tangential coordinate and  $(R + \sigma)$  is the distance from the origin. With our conventions, both scalar fields have units of energy, as does  $R$ .
- [31] It is also worth pointing out a similar mechanism may be at work in the Standard Model. With the measured mass of the Higgs boson at 125 GeV, the Higgs potential develops an instability due to renormalization group flow at an high scale, below the Planck scale [29]. This hints that landscape statistics push the parameters of the Higgs potential to a point of marginal stability before new physics takes over.